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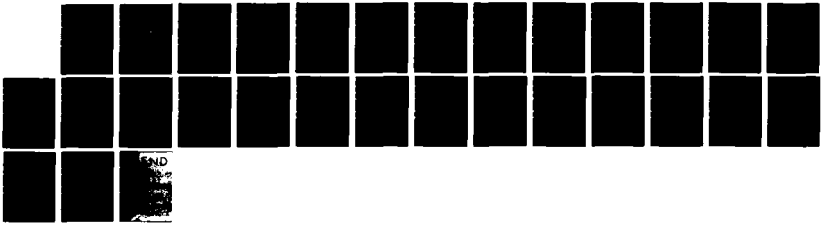

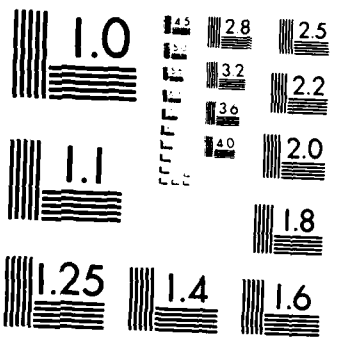

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Pari-Mutuel as a System of

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by

Guillermo Owen

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A bettor is faced with the problem of choosing optimal bets, given his (subjective) probabilities of outcomes of an experiment and the payoff odds on these outcomes. Conversely, a bookie (pari-mutuel system), faced by several bettors with different subjective probabilities, has the problem of choosing payoff odds so as to avoid the risk of loss. It is shown that, under certain reasonably broad conditions on the several

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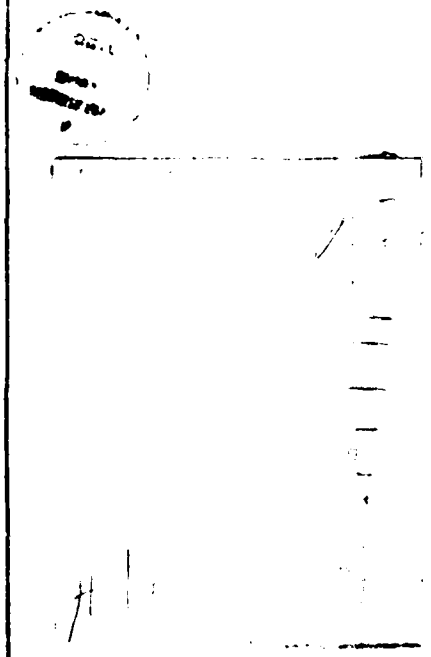
bettors' utilities and subjective probabilities, such an equilibrium set of payoff odds always exists. Some examples are worked out in detail.



Pari-Mutuel as a System of Aggregation of Information

1. The Individual Bettor

Let us consider the problem faced by a bettor at a race track. There are n horses, with (subjective) probabilities $p_1, p_2, \ldots, p_n$ of winning. He has A dollars to bet on these horses, and a bet of x dollars on horse j will return x/q_j dollars if that horse wins. (The x/q_j includes the bettor's original x dollar bet.) It is assumed that the p_j and q_j satisfy the standard conditions

$$(1) \qquad \sum_{j=1}^{n} p_j = 1 \; ; \; p_j \geq 0$$

and

$$(2) \qquad \sum_{j=1}^{n} q_j = 1 \; ; \; q_j \geq 0$$

Condition (1) means the bettor's subjective probabilities are consistent; condition (2) means the payoff odds are fair. (Though, in fact, they seldom are, and most bookies will normally announce odds such that $\sum q_j$ is substantially greater than 1, i.e., they pay less than a "fair" system would.)

The bettor has a utility function u for money; it will be assumed that u is monotone non-decreasing and continuous. The bettor's problem, then, is to choose his bets, $x_1, \ldots, x_n$, so as to maximize his expected utility, given as

$$(3) \qquad F(x_1, \ldots, x_n) = \sum_{j=1}^{n} p_j \, u\left(\frac{x_j}{q_j} - B + A\right).$$

where

$$B = \sum_{j=1}^{n} x_j \tag{4}$$

subject to

$$B \leq A \tag{5}$$

$$x_j \geq 0 \quad . \tag{6}$$

The first thing we notice is that, assuming the fairness conditions (2), the bettor might as well set $B = A$, i.e. bet all his available funds. In fact, suppose we had $B < A$. We could then set $\epsilon = A - B$, and

$$x_j' = x_j + \epsilon q_j \qquad j = 1,\ldots,n \quad .$$

In this case,

$$B' = \sum x_j' = B + \epsilon = A$$

and, moreover,

$$F(x_1,\ldots, x_n) = \sum p_j \, U\left(\frac{x_j}{q_j} + \epsilon\right)$$

$$F(x_1',\ldots x_n') = \sum p_j \, U\left(\frac{x_j + \epsilon q_j}{q_j}\right)$$

so that $F(x') = F(x)$, *i.e.* the bettor can do at least as well with bets x_j' such that $B' = A$ as with any other bets with $B \leq A$.

We simplify the problem, then, to one of maximizing

$$F(x_1,\ldots, x_n) = \sum_j p_j \, U\left(\frac{x_j}{q_j}\right) \tag{7}$$

subject to

$$\sum_j x_j = A \tag{8}$$

$$x_j \geq 0 \quad . \tag{9}$$

Assuming differentiability of u, the first-order conditions for optimality will be

$$\frac{p_j}{q_j} \; u'\left(\frac{x_j}{q_j}\right) = \lambda \qquad \text{if} \quad x_j > 0 \tag{10a}$$

$$\frac{p_j}{q_j} \; u'(0) \leq \lambda \qquad \text{if} \quad x_j = 0 \tag{10b}$$

where λ is a Lagrange multiplier representing the marginal utility of money.

In case u is not differentiable at the point x_j/q_j, conditions (10) must be modified, in terms of the right-hand and left-hand derivatives of u to give

$$\frac{p_j}{q_j} \; u'\left(\frac{x_j}{q_j} +\right) \leq \lambda \leq \frac{p_j}{q_j} \; u'\left(\frac{x_j}{q_j} -\right) \tag{11}$$

Now (10a) can be rewritten as

$$p_j \; u'\left(\frac{x_j}{q_j}\right) = \lambda q_j .$$

In the simplest case, all x_j are positive, so that (10a) holds for all j. Adding with respect to j, we have, by (2)

$$\lambda = \sum_j p_j \; u' \; \frac{x_j}{q_j} \tag{12}$$

so that λ is simply the expected value of u'.

More generally, of course, (12) does not hold for all j, and

so we can only state that λ is at least equal to the expected value of U'.

In case U is concave, the first-order conditions (10) are sufficient for optimality. We rewrite these as

$$U'\left(\frac{x_j}{q_j}\right) = \frac{\lambda q_j}{p_j} \qquad \text{if } x_j > 0$$

$$U'(0) \leq \frac{\lambda q_j}{p_j} \qquad \text{if } x_j = 0 \tag{14}$$

Using the fact that U' is monotone non-increasing (for concave U), we obtain the interesting result

$$\frac{x_j}{q_j} \geq \frac{x_k}{q_k} \quad \text{whenever} \quad \frac{p_j}{q_j} > \frac{p_k}{q_k} \tag{15}$$

with the stronger result that, for strictly concave U, (15) holds even if the second inequality is loose.

Thus, a discrepancy between the bettor's subjective probabilities and the payoff odds leads the bettor to bet so that his conditional winnings will be greater for horses for which the ratio p_j/q_j is greater, and conversely.

Conditions (13) and (14) are meaningful if both p_j and q_j are positive. In case $p_j = 0$, $q_j > 0$, it is easily seen that optimality requires $x_j = 0$, i.e. never bet on a horse which (subjectively) has no chance of winning. It is not clear what happens if $q_j = 0$, though in practice it is difficult to imagine a situation in which infinite odds were offered. In case $p_j = q_j = 0$, we imagine the bettor will still set $x_j = 0$; in case $p_j > q_j = 0$, however, we seem to reach some sort of contradiction. We note, then that $q_j = 0$ leads to contradictions which would best

be avoided; among other things, the payoff functions are discontinuous or fail to exist here.

In case U is strictly concave, we may use the inverse function $w = (U')^{-1}$ and (13) - (14) now take the form

$$x_j = q_j \, w\left(\frac{\lambda q_j}{p_j}\right) \quad \text{if} \quad w\left(\frac{\lambda q_j}{p_j}\right) > 0 \tag{16}$$

$$x_j = 0 \quad \text{if} \quad w\left(\frac{\lambda q_j}{p_j}\right) \leq 0 \tag{17}$$

Condition (8) can be restated as

$$A = \sum{}' \, q_j \, w\left(\frac{\lambda q_j}{p_j}\right) \tag{18}$$

where the prime on the summation symbol means that it should consider only those j such that (16) holds, i.e. such that

$$\lambda q_j < p_j \, U'(0) \tag{19}$$

The right side (18) can be seen to be a monotone non-increasing function of λ and thus (18) can be solved, numerically or analytically, for λ. This presumably solves the single bettor's problem.

2. The Equilibrium Odds

In general, bookies tend to be risk-averse and seek to set payoff odds in such a way as to eliminate the possibility of loss. Of course, a bookie is not bound by the fairness condition (2), so that, in practice, the sum of the q_j is greater than 1. If (2) were to be enforced, however (perhaps under cutthroat competition among bookies), the bookie could only eliminate the risk of loss if the amounts bet on the several horses were proportional to the q_j, i.e. if

$$b_j = q_j C \tag{20}$$

where b_j is the total amount bet on horse j (by all bettors) and C is the total amount of all wagers.

If there is only one bettor, it is easy to see that this can be accomplished by setting $q_j = p_j$. For then $x_j = q_j A$ will satisfy conditions (10) (with $\lambda = U'(A)$). In case U is strictly concave, moreover, this is the bettor's unique optimum, so that $q_j = p_j$ gives rise to an equilibrium. (Clearly, with one bettor, $b_j = x_j$ and $C = A$).

If there are two or more bettors, the bookie must look for some way of combining the several bettors' subjective probabilities so as to avoid risk. At a race track, this is normally accomplished by a pari-mutuel system, which simply sets $q_j = b_j/C$, so that (20) is automatically achieved, after the amounts bet are known. In effect, the players bet against each other, with the track as intermediary. This has the disadvantage - from the players' point of view - that bets are made with only partial knowledge of the

payoff odds. Thus, a player might well feel he would have changed his bets, had he known the true payoff odds in advance. Of course such a change would in turn cause the q_j to change, leading to a further change in bets, et sic ad infinitum, or at least until some equilibrium is reached. The question is whether such an equilibrium exists.

Assume, then, m bettors. Bettor i $(i = 1,\ldots, m)$ has a subjective probability distribution $(p_{i1}, p_{i2},\ldots, p_{in})$ satisfying $p_{ij} \geq 0$, and

$$\sum_{j=1}^{n} p_{ij} = 1 \quad .$$

This same bettor has a sum of money, A_i, available for betting, and a utility function for money, U_i. If the odds are posted as $(q_1, q_2,\ldots,q_n)$, then each bettor will choose $(x_{i1}, x_{i2},\ldots, x_{in})$ so as to maximize his expected utility, as discussed above. Total bets on horse j are then

$$b_j = \sum_{i=1}^{m} x_{ij} \tag{21}$$

and the total amount bet on all horses is

$$C = \sum_{i=1}^{m} A_i = \sum_{j=1}^{n} b_j \quad . \tag{22}$$

There will be an equilibrium if (20) holds for all j.

As was mentioned above, difficulties arise if $q_j = 0$ for any j. We will therefore try to avoid this, and will specifically rule out such equilibria. We make then the following assumption.

Assumption Z. For every j, there is some i such that $p_{ij} > 0$.

We prove the existence of equilibrium under the further assumption that the utility functions are strictly concave. Essentially, this uses a fixed-point theorem. Some care must however be used to avoid the possibility that the fixed point lies on the boundary of the simplex.

Theorem 1. Suppose Assumption z holds, and suppose moreover that all the utility functions are strictly concave. Then there is an equilibrium n-tuple of payoff odds, $q_j^* > 0$.

Proof: Let Q be the unit n-simplex, i.e. the set of vectors $(q_1, \ldots, q_n)$ satisfying (2). Let Q° be the interior of Q (the set of q with all components positive) and let ∂Q be the boundary of Q (the set of q with at least one $q_j = 0$.)

For $q \in Q^\circ$, consider bettor i's optimal choice of bets. As discussed above, it cannot be optimal for him to bet on a horse with no chance of winning, so his bets must satisfy, not just (8) and (9), but also the condition $x_{ik} = 0$ whenever $p_{ik} = 0$.

Restricted to that set, bettor i's expected utility,

$$F_i(x_i, q) = \sum_{j=1}^{n} p_{ij} U_i\left(\frac{x_{ij}}{q_i}\right)$$

is strictly concave, and so has a unique maximizing vector, $x_i^*(q)$. Since F is continuous for all x_i and all $q \in Q^\circ$, it will follow that $x_i^*(q)$ is continuous for $q \in Q^\circ$.

Let, now,

$$b^*(q) = \sum_{i=1}^{m} x_i^*(q)$$

Then b^* is a continuous mapping from Q° into $\mathbb{R}^n$. Let, finally,

$$J(q) = \left\{ j \;\middle|\; \frac{b_j^*(q)}{q_j} = \max_k \frac{b_k^*(q)}{q_k} \right\}$$

Clearly, J assigns to each $q \in Q^o$ a non-empty subset of $N = \{1, 2, \ldots, n\}$. By the continuity of b^*, J is upper semi-continuous.

Next, for $q \in \partial Q$, define

$$J(q) = \left\{ j \;\middle|\; q_j = 0 \right\} .$$

Since $q \in \partial Q$, $J(q)$ is non-empty here also. Trivially, it is upper semi-continuous if restricted to ∂Q.

In this way, the mapping J is defined over the entire simplex Q. We wish to show it is upper semi-continuous, i.e. if $q \longrightarrow \hat{q}$ and $j \in J(q)$, then $j \in J(\hat{q})$.

In this, we can dispense with the case in which $\hat{q} \in Q^o$, since such $\hat{q}$ can only be approached through $q \in Q^o$, and we know J, restricted to Q^o, is semi-continuous. Similarly, we can dispense with the case that $q \longrightarrow \hat{q}$, with all q and $\hat{q} \in \partial Q$, since we know J, restricted to ∂Q, is semi-continuous.

It remains to consider the case in which $q \longrightarrow \hat{q}$ with $q \in Q^o$ and $\hat{q} \in \partial Q$. Let $K = J(\hat{q})$. We must show that, for q sufficiently close to $\hat{q}$, $J(q) \subset K$

Take some (fixed) $k \in K$: we have $\hat{q}_k = 0$. By assumption Z, there is some bettor, h, with $p_{hk} > 0$. Keeping h fixed, let $L(q)$ be the set of all j for which p_{hj}/q_j is maximal.

Suppose $\hat{q}_j > 0$. As $q \longrightarrow \hat{q}$, the ratio p_{hk}/q_k increases without bound, whereas p_{hj}/q_j approaches the finite limit, $p_{hj}/\hat{q}_j$. Thus $j \notin L(q)$, and we conclude there exists $\varepsilon_1 > 0$ such that, if

$|q - \hat{q}| < \varepsilon_1$, $L(q) \subset K$.

Let r be the minimum of all $\hat{q}_j$ such that $\hat{q}_j > 0$. Let $\varepsilon_2 = r/2$. Then, for all q such that $|q - \hat{q}| < \varepsilon_2$ and all $j \notin K$, we will have $q_j > r/2$.

Let s be the minimum of all p_{hj} such that $\hat{q}_j = 0$ and $p_{hj} > 0$. Since U_h is strictly increasing and concave, we know $U_h'(y+) > 0$ for all y. Set, then,

$$\varepsilon_3 = \frac{rs}{2} U_h' \left(\frac{2C}{r}\right) \Big/ U_h' \left(\frac{1}{2}\right)$$

Finally, let $\varepsilon_4 = \dfrac{r}{4nC}$

Let now ε be the smallest of ε_1, ε_2, ε_3, and ε_4. Assume $q \in Q^o$, $|q - \hat{q}| < \varepsilon$. We will show $J(q) \subset K$.

Let $x_h^*(q)$ be bettor h's optimal response to the payoff odds q, and suppose $x_{hj}^* > q_j/2$ for some $j \notin K$. Let $\ell \in L(q)$. Since $|q - \hat{q}| < \varepsilon_1$, $\ell \in K$. Moreover, $j \notin L(q)$, so

$$\frac{p_{h\ell}}{q_\ell} > \frac{p_{hj}}{q_j}$$

and hence, by (15),

$$\frac{x^*_{h\ell}}{q_\ell} \geq \frac{x^*_{hj}}{q_j}$$

Thus $x^*_{h\ell}$ and x^*_{hj} are both positive, and so we can apply (13) to get

$$U_h' \left(\frac{x^*_{h\ell}}{q_\ell}\right) = \frac{q_\ell \, p_{hj}}{q_j \, p_{h\ell}} \, U_h' \left(\frac{x^*_{hj}}{q_j}\right)$$

Now, since $|q - \hat{q}| < \varepsilon$, we will have $q_j > r/2$, $q_\ell < \varepsilon_3$, and $p_{h\ell} \geq s$.

Also, $p_{hj} \leq 1$. Finally, $x^*_{hj}/q_j > \frac{1}{2}$, so by monotonicity $U_h'\left(\frac{x^*_{hj}}{q_j}\right) \leq U_h'\left(\frac{1}{2}\right)$. Thus

$$U_h'\left(\frac{x^*_{h\ell}}{q_\ell}\right) < \frac{2\varepsilon}{rs}\, U_h'\left(\frac{1}{2}\right)$$

and, using the definition of ε_3,

$$U_h'\left(\frac{x^*_{h\ell}}{q_\ell}\right) < U_h'\left(\frac{2C}{r}\right) \quad .$$

By the monotonicity of U', this gives

$$\frac{x^*_{h\ell}}{q_\ell} \geq \frac{2C}{r} \quad .$$

Clearly, $b^*_\ell \geq x^*_{h\ell}$, and so $b^*_\ell/q_\ell \geq \frac{2C}{r}$.

On the other hand, for any $k \notin K$, $b^*_k \leq C$ and $q_k > r/2$. Thus

$$\frac{b^*_k}{q_k} < \frac{2C}{r} \leq \frac{b^*_\ell}{q_\ell}$$

and we see that $k \notin J(q)$. We conclude that $J(q) \subset K$.

Suppose, on the other hand, there is no j, $j \notin K$, with $x^*_j > q_j/2$. In this case,

$$\sum_{j \notin K} x^*_{hj} \leq \frac{1}{2} \sum_{j \notin K} q_j < \frac{1}{2}$$

and so

$$\sum_{j \in K} x^*_{hj} > A_h - \frac{1}{2} \geq \frac{1}{2}$$

Thus there is some $\ell \in K$ with $x^*_{h\ell} > \frac{1}{2n}$.

Since $|q - \hat{q}| < \varepsilon_4$ and $q_\ell = 0$, we have $q_\ell < \varepsilon_4$ and so

$$\frac{b^*_\ell}{q_\ell} \geq \frac{x^*_{h\ell}}{q_\ell} > \frac{1}{2n\,\varepsilon_4} = \frac{2C}{r}$$

Once again $b_j^*/q_j < 2C/r$ for all $j \notin K$, and so $b_j^*/q_j < b_\ell^*/q_\ell$. Thus we conclude once that $J(q) \subset K$.

We see then that J is an upper semi-continuous mapping, assigning a non-empty subset of N to each $q \in Q$. Define, now, for $S \subset N$,

$$\phi(S) = \left\{ q \;\middle|\; \begin{array}{l} q \in Q \\ q_j = 0 \text{ if } j \notin S \end{array} \right\}.$$

Clearly, ϕ is an upper semi-continuous mapping from the subsetsof N to Q. The composition, $\Phi = \phi o J$, is then an upper semi-continuous mapping assigning a non-empty, closed convex subset of Q to each $q \in Q$. By the Kakutani fixed-point theorem, such a mapping must have a fixed point, i.e. there is $q^* \in Q$, such that $q^* \in \Phi(q^*)$.

Clearly $q^* \notin \partial Q$ since, for $q \in \partial Q$, $J(q)$ consists of those indices j with $q_j = 0$, and so $\Phi(q)$ will consist of those vectors $z \in Q$ such that $z_j = 0$ whenever $q_j > 0$. Thus $q^* \in Q^o$. But, if $q^* \in Q^o$, the only $S \subset N$ such that $q^* \in \phi(S)$ is N itself, i.e. $J(q^*) = N$. This means that $\dfrac{b_j^*(q^*)}{q_j^*}$ is equal for all j, and this will mean that

$$b_j^*(q^*) = q_j^* C.$$

Thus q^* is the desired equilibruim odds vector.

The hypothesis of Theorem 1 - mainly, strict concavity - is overly restrictive. We can weaken it to require only (weak) concavity together with strict monotonicity of the utility functions.

Assume, then, that the U_i are merely concave functions of money. For $t \geq 0$, define

$$w_i(x,t) = U_i(X) - t\, e^{-x}$$

Then $w_i(x,0) = U_i(x)$. We have, however,

$$w_i'(x,t) = U_i'(x) + t\, e^{-x}$$

$$w_i''(x,t) = U_i''(x) - t\, e^{-x}$$

(where, in all cases, the primes denote differentiation with respect to x) and so we find w_i is strictly concave and monotone in x for each $t > 0$.

Suppose, then, that each bettor's utility function $U_i(x)$ is replaced by the strictly concave $w_i(x;t)$. For each $t > 0$, there is an equilibrium n-tuple $q^*(t)$. As $t \longrightarrow 0$, these $q^*(t)$ will have an accumulation point, q^{**}. The only difficulty is that, though all $q^*(t) \in Q^\circ$, q^{**} could conceivably belong to ∂Q. Theorem 2 says this will not happen.

Theorem 2. Assume all the functions U_i are concave, strictly monotone increasing, and suppose Assumption Z holds. Then there exists an equilibrium n-tuple $q^{**} \in Q^\circ$.

Proof: As discussed above, let $q^*(t)$ be the equilibrium obtained with the utility functions $w_i(x;t)$. Let $t \longrightarrow 0$; by the compactness of Q, the points $q^*(t)$ will have some accumulation point $q^{**} \in Q$. We must show $q^{**} \notin \partial Q$.

Suppose, then, $\hat{q} \in \partial Q$. We will show that, if $|q - \hat{q}|$ and t are sufficiently small, $q \neq q^*(t)$.

In fact, if $q = q^*(t)$, then $J_t(q) = N$, where $J_t(q)$ is the set J, as described above, with U_i replaced by $w_i(x,t)$.

Define ε_1, ε_2, ε_3, ε_4, as in the proof of Theorem 1, and let $\varepsilon_5 = \varepsilon_3/2$. Let $\delta < U_i'(\frac{1}{2})$.

Suppose, now, that ε is the minimum of ε_1, ε_2, ε_4, ε_5, that $|q - \hat{q}| < \varepsilon$, and that $0 < t < \delta$. We have, now,

$$w_h'\left(\frac{2C}{r}, t\right) = U_h'\left(\frac{2C}{r}\right) + t\, e^{-\frac{2C}{r}} > U_h'\left(\frac{2C}{r}\right)$$

$$w_h'\left(\frac{1}{2}, t\right) = U_h'\left(\frac{1}{2}\right) + t\, e^{-\frac{1}{2}} < U_h'\left(\frac{1}{2}\right) + \delta < 2U_h'\left(\frac{1}{2}\right)$$

and so

$$\varepsilon_4 = \frac{\varepsilon_3}{2} = \frac{rs}{4} \cdot \frac{U_h'\left(\frac{2C}{r}\right)}{U_h'\left(\frac{1}{2}\right)} < \frac{rs}{2} \frac{w_h'\left(\frac{2C}{r}, t\right)}{w_h'\left(\frac{1}{2}, t\right)}$$

The proof then proceeds as in that of Theorem 1, leading to the conclusion that

$$J_t(q) \subset K \neq N$$

and so $q \neq q^*(t)$. Thus $\hat{q} \neq q^{**}$, i.e. $q^{**} \in Q^o$ as desired.

It is easy to see that, if $q^*(t)$ is an equilibrium for each t, then q^{**} will be an equilibrium for $t = 0$, i.e. for the utility functions U_i.

It is possible to weaken the conditions on the utility functions further - so that they are not strictly increasing, but only if we strengthen condition z. Consider then

Assumption Y. For all i and j, $p_{ij} > 0$.

Theorem 3. Assume all the utility functions are concave and non-decreasing, and suppose Assumption Y holds. Then there is an equilibrium $q^{**} \in Q^o$.

Proof: We consider first the trivial case in which, for each bettor i, $U_i(x)$ is maximized at $x = A_i$ (or less). In this case, it is clear that, for any $q \in Q^o$, $x_{ij} = A_i q_j$ will be optimal for bettor i, and so every $q \in Q^o$ will give an equilibrium.

Suppose, then, that for some bettor, say bettor 1, U, is not maximized at A_1. Then $U_1'(A_1,+) > 0$, and we will set

$$r = \frac{U_1'(A_1 +)}{U_1'(A_1 -)} \quad .$$

Clearly $0 < r \leq 1$.

As before, we replace the utility functions $U_i(x)$ by $w_i(x,t) = U_i(x) - t\,e^{-x}$, and consider the equilibria $q^*(t)$ obtained in this manner. (Theorem 1 guarantees their existence.) As $t \longrightarrow 0$, these $q^*(t)$ have an accumulation point $q^{**} \in Q$. If $q^{**} \in Q^o$, it is the desired equilibrium. We must show $q^{**} \notin \partial Q$.

In fact, assume $\hat{q} \in \partial Q$, and let K be the set of all k such that $\hat{q}_k = 0$. Let $\varepsilon > 0$ be such that, if $|q - \hat{q}| < \varepsilon$,

$$\frac{p_{ij}}{q_j} < \frac{r}{2} \frac{p_{ik}}{q_k}$$

for all i, all $k \in K$, and $j \notin K$. (This can be done since all $p_{ik} > 0$.)

By (15), and since $r \leq 1$, we have

$$\frac{x_{ik}}{q_k} \geq \frac{x_{ij}}{q_j} \tag{24}$$

for all i, all $j \notin K$, $k \in K$.

Since (24) holds for all i, the equilibrium condition (20) can hold only if (24) holds as an equation throughout. This will mean

$$\frac{x_{1j}}{q_j} = A_1 \qquad j = 1, \ldots, n$$

Then, for $k \in K$, $j \notin K$,

$$\frac{p_{1j}}{q_j} < \frac{r}{2} \frac{p_{1k}}{q_k}$$

and, by (11)

$$\frac{w_1'(A_1+;\ t)}{w_1'(A_1-;\ t)} < \frac{r}{2}$$

or

$$\frac{U_1'(A_1+) + t\,e^{-A_1}}{U_1'(A_1-) + t\,e^{-A_1}} < \frac{r}{2}$$

so that, since $r \leq 1$,

$$\frac{U_1'(A_1+)}{U_1'(A_1-)} < \frac{r}{2}$$

and this is a contradiction. Thus q cannot be $q^*(t)$ for any t, and therefore $\hat{q} \neq q^{**}$. We conclude that $q^{**} \in Q^0$, and is the desired equilibrium.

3. Examples

We consider here several examples. The first three consider some "reasonable" utility functions; the last shows that the conditions of Theorems 2 and 3 cannot be further weakened.

(a) Logarithmic utility

Assume that each bettor has a utility function

$$U_i(x) = \log(K_i + x)$$

where K_i is a parameter, representing perhaps player i's reserves. In this case, the optimability conditions (10) - (11) take the form

$$\frac{p_{ij}}{K_i q_j + x_{ij}} = \lambda_i \quad \text{if } x_{ij} > 0$$

or, equivalently,

(25) $$x_{ij} = \frac{p_{ij}}{\lambda_i} - K_i q_j \quad \text{if positive}$$

(26) $$x_{ij} = 0 \quad \text{if} \quad p_{ij} \leq K_i q_j \lambda_i .$$

In the simplest case, when all $x_{ij} > 0$, this will give us

$$A_i = \sum_{j=1}^{n} x_{ij} = \frac{1}{\lambda_i} - K_i$$

and so

$$\lambda_i = \frac{1}{A_i + K_i} \quad .$$

Substituting in (25), this gives us

$$x_{ij} = (A_i + K_i)\, p_{ij} - K_i q_j \quad .$$

To look for the equilibrium, we have, from (20) - (21)

$$\sum_{i=1}^{m} x_{ij} = q_j^* \, C.$$

Thus

$$\sum_i (A_i + K_i) p_{ij} - q_j^* \sum_i K_i = q_j^* \, C$$

so

(27)
$$q_j^* = \frac{\sum_i (A_i + K_i) p_{ij}}{C + \sum_i K_i}$$

But $C = \sum A_i$, and we see that q_j^* is then simply a weighted average, with weights $A_i + K_i$, of the several subjective probabilities p_{ij}.

(b) Exponential utilities

Consider, next, the exponential utility functions

$$U_i(x) = -e^{-\alpha_i x}$$

where α_i is a parameter, measuring, in some sense, bettor i's risk aversion. (Essentially, $1/\alpha_i$ can be thought of as representing the sum of money which i would be "hurt" by losing.)

In this case, the optimality conditions (10)-(11) take the form

$$\alpha_i p_{ij} \exp\left\{-\frac{\alpha_i x_{ij}}{q_j}\right\} = \lambda_i q_j \quad \text{if } x_{ij} > 0$$

which reduces to

$$\alpha_i x_{ij} = q_j (\log \alpha_i - \log \lambda_i) + q_i \log \frac{p_{ij}}{q_j} \quad . \tag{28}$$

if this (right side) is positive, and $x_{ij} = 0$ otherwise (i.e. $x_{ij} = 0$ if $\alpha_i p_{ij} < \lambda_i q_j$).

Assuming, again, that all $x_{ij} > 0$, summation with respect to j gives us

$$\alpha_i A_i = \log \alpha_i - \log \lambda_i + \sum_j q_j \log \frac{p_{ij}}{q_j}$$

so that, substituting in (28),

$$\frac{x_{ik}}{q_k} = \frac{1}{\alpha_i}\left[\log \frac{p_{ik}}{q_k} - \sum_j q_j \log \frac{p_{ij}}{q_j}\right] + A_i$$

For the equilibrium odds, we add with respect to i, obtaining

$$C = \sum_i \frac{1}{\alpha_i}\left[\log \frac{p_{ik}}{q_k} - \sum_j q_j \log \frac{p_{ij}}{q_j}\right] + C$$

or

$$\sum_i \frac{1}{\alpha_i} \log \frac{p_{ik}}{q_k} = \sum_i \sum_j \frac{1}{\alpha_i} q_j \log \frac{p_{ij}}{q_j} \quad .$$

The right side of this last expression is independent of k, and so

$$\sum_i \frac{1}{\alpha_i} \log p_{ik} - \log q_k \sum_i \frac{1}{\alpha_i} = \gamma$$

where γ is independent of k. Then

$$\log q_k^* = \frac{\sum r_i \log p_{ik} - \gamma}{\sum r_i}$$

or

$$(29) \qquad q_k^* = \Gamma \left\{ \prod_{i=1}^{n} p_{ik}^{r_i} \right\}^{\frac{1}{r_1+\ldots+r_n}}$$

where $r_i = \frac{1}{\alpha_i}$, and Γ is constant. Thus in this case q_k^* is proportional to a weighted geometric mean, with weights $\frac{1}{\alpha_i}$, of the probabilities p_{ik}.

(c) Linear utility.

Yet another possibility is to equate utility with money. This case was treated in detail by Gale and Eisenberg (1959) and so we will merely refer the reader to that interesting article.

(d) A counter-example.

Let us consider a two-horse, two-bettor situation. Bettor 1 is certain horse 1 will win whereas bettor 2 feels the race is up for grabs. They have equal capital. Thus

$$p_{11} = 1 \qquad p_{12} = 0$$
$$p_{21} = \frac{1}{2} \qquad p_{22} = \frac{1}{2} \quad .$$

$$A_1 = A_2 = 1$$

Bettor 1's utility function does not much matter - he will bet all his money on horse 1. Bettor 2 has a utility function

$$U_2(x) = \begin{cases} x & \text{if} \quad x \leq \frac{3}{2} \\ \frac{3}{2} & \quad x \geq \frac{3}{2} \end{cases} \quad .$$

It is easily seen that, if $q_1 < \frac{1}{2}$, then bettor 2 will choose

$$x_{21} = \frac{3q_1}{2} \qquad x_{22} = 1 - \frac{3q_1}{2}$$

whereas, if $q_1 > \frac{1}{2}$, then 2 will choose

$$x_{21} = 1 - \frac{3q_2}{2} \qquad x_{22} = \frac{3q_2}{2}$$

For $q_1 = q_2 = \frac{1}{2}$, we would have $\frac{1}{4} \le x_{21} \le \frac{3}{4}$, $x_{22} = 1 - x_{21}$. Since $x_{11} = 1$, $x_{12} = 0$ whatever q is, we will then have

$$b_1^*(q) = 1 + \frac{3q_1}{2} \quad \text{if } q_1 < \frac{1}{2}$$

$$b_1^*(q) = 2 - \frac{3q_2}{2} = \frac{1}{2} + \frac{3q_1}{2} \quad \text{if } q_1 > \frac{1}{2}$$

$$\frac{5}{4} \le b_1^*(q) \le \frac{7}{4} \quad \text{if } q_1 = \frac{1}{2} \quad .$$

For an equilibrium, we must have $b_1^*(q) = 2q_1$ (since $C = A_1 + A_2 = 2$). But, from the above, we see this holds only if $q_1 = 1$. But this leads to the undesirable discontinuity on the boundary of the simplex, and we must conclude that there is no equilibrium for this situation.

APPENDIX

Concavity and Convexity

A function f from $\mathbb{R}^m$ to $\mathbb{R}$ is said to be concave if, for every $x, y \in \mathbb{R}^m$, and $0 \leq \lambda \leq 1$

(30) $$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y).$$

It is strictly concave if strict inequality holds in (30) whenever $x \neq y$ and $0 < \lambda < 1$. A function f is convex [strictly convex] if $-f$ is concave [strictly concave].

A set $S \subset \mathbb{R}^m$ is convex if, for any $x, y \in S$ and $0 \leq \lambda \leq 1$,

$$\lambda x + (1-\lambda) y \in S.$$

Generally, if f is a concave function, for any q, the set

$$S_q = \{x \mid f(x) \geq q\}$$

is convex. In particular, the set of all x which maximize $f(x)$ is convex (though it may be empty).

If a function f is strictly concave, it need not have a maximum. If there is a maximum, however, the maximizing point is unique.

If $f: \mathbb{R} \longrightarrow \mathbb{R}$ is concave, it will be differentiable almost everywhere in its domain. Even when not differentiable, however, f has both right and left derivatives, $f'(x+)$ and $f'(x-)$. The derivative is monotone non-increasing, satisfying

31(a) $$f'(x+) \leq f'(x-) \quad \text{for all } x$$

31(b) $$f'(x-) \leq f'(y+) \quad \text{if } y < x.$$

If f is strictly concave, its derivative is strictly monotone, satisfying 31(a) and satisfying 31(b) with strict inequality.

Upper Semi-Continuity

Let X, Y be topological spaces. A set-valued mapping from X to Y is a mapping ϕ which assigns, to each $x \in X$, a subset $\phi(x) \subset Y$. It is a correspondence if $\phi(x) \neq \emptyset$ for all $x \in X$.

The set-valued mapping ϕ is said to be upper semi-continuous if, whenever $x_n \longrightarrow x^*$, $y_n \in \phi(x_n)$, and $y_n \longrightarrow y^*$ then $y^* \in \phi(x^*)$.

Theorem. Let f be a continuous real-valued function defined on the product space $X \times Y$. Define for $x \in X$,

$$\phi(x) = \left\{ y \mid f(x,y) = \max_{t \in y} f(x,t) \right\}$$

then ϕ is an upper semi-continuous set-valued mapping from X to Y. If Y is compact and non-empty, then ϕ is also a correspondence.

Kakutani's Fixed-Point Theorem.

Let X be a simplex in $\mathbb{R}^n$, and let ϕ be an upper semi-continuous correspondence from X to X, such that, for all x, $\phi(x)$ is compact and convex. Then there is some $x^* \in X$ such that $x^* \in \phi(x^*)$.

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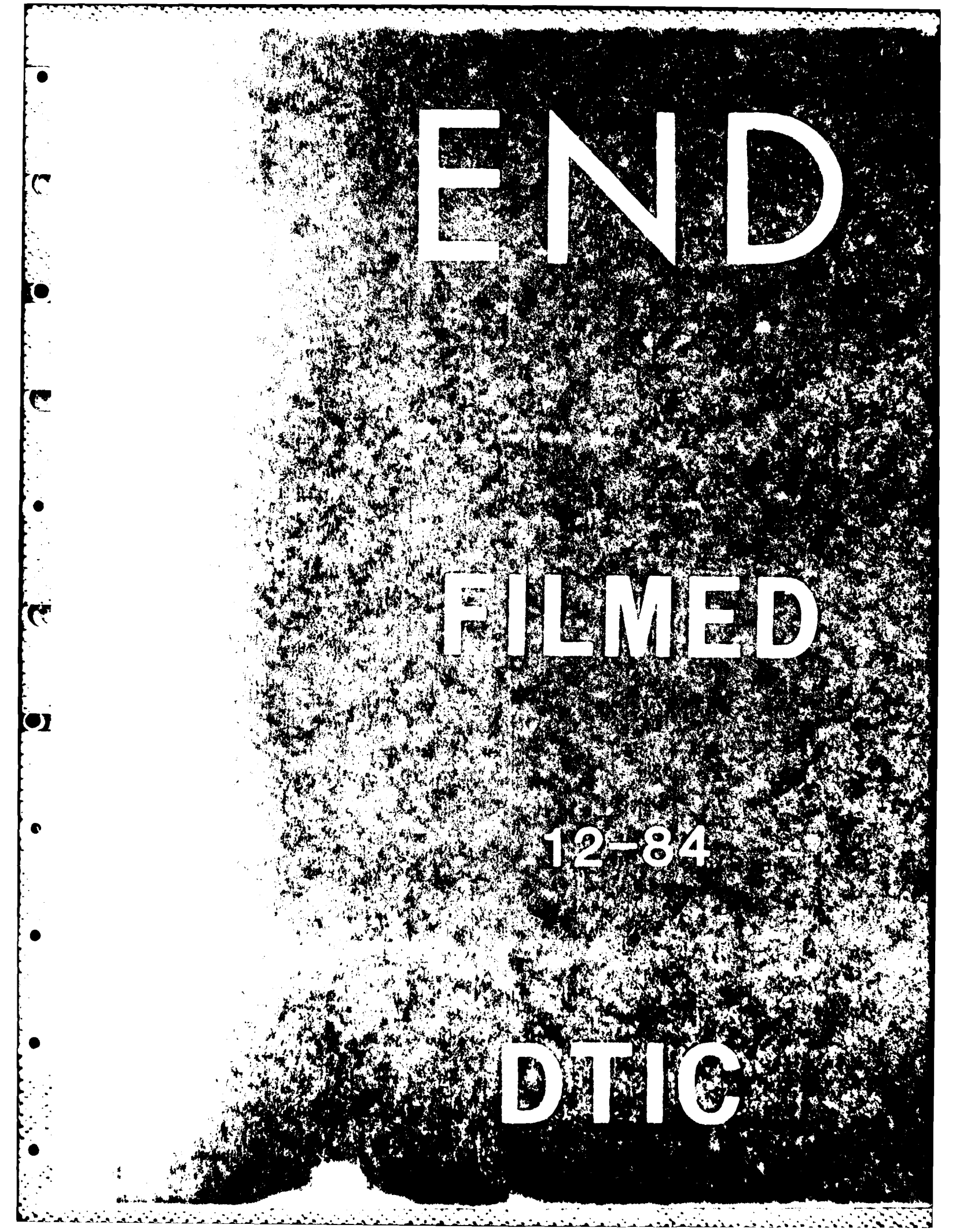
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